Using fractional delay to control the magnitudes and phases of integrators and differentiators

M.A. Al-Alaoui

Abstract: The use of fractional delay to control the magnitudes and phases of integrators and differentiators has been addressed. Integrators and differentiators are the basic building blocks of many systems. Often applications in controls, wave-shaping, oscillators and communications require a constant 90° phase for differentiators and −90° phase for integrators. When the design neglects the phase, a phase equaliser is often needed to compensate for the phase error or a phase lock loop should be added. Applications to the first-order, Al-Alaoui integrator and differentiator are presented. A fractional delay is added to the integrator leading to an almost constant phase response of −90°. Doubling the sampling rate improves the magnitude response. Combining the two actions improves both the magnitude and phase responses. The same approach is applied to the differentiator, with a fractional sample advance leading to an almost constant phase response of 90°. The advance is, in fact, realised as the ratio of two delays. Filters approximating the fractional delay, the finite impulse response (FIR) Lagrange interpolator filters and the Thiran allpass infinite impulse response (IIR) filters are employed. Additionally, a new hybrid filter, a combination of the FIR Lagrange interpolator filter and the Thiran allpass IIR filter, is proposed. Methods to reduce the approximation error are discussed.

1 Introduction

Rabiner and Steiglitz in [1] noted that introducing a half sample delay to an ideal differentiator does not change the magnitude response; however, the phase response will become linear [2]. The magnitude and phase responses of the ideal differentiator are shown in Fig. 1a and b, respectively. The Nyquist frequency is normalised to 1, and the sampling frequency is twice the Nyquist frequency. The ideal differentiator has a constant phase response of 90° (Fig. 1b). We can clearly see the discontinuity of the phase at the Nyquist frequency. An ideal half sample delay has a magnitude of unity (Fig. 1c) and a linear phase (Fig. 1d). Delaying the response of the ideal differentiator by a half sample results in a linear phase as shown in Fig. 1f. (The magnitude is not affected when the delay is ideal as shown by comparing Fig. 1e with Fig. 1a.) This procedure is used in [1] to overcome the problem of phase discontinuity at the Nyquist frequency.

In contrast, Tseng [3] used fractional delay to improve the magnitude response of the Simpson integrator. The transfer function of the Simpson integrator is shown as

\[ H_S(z) = \frac{T}{3} \frac{1 + 4z^{-1} + z^{-2}}{1 - z^{-2}} \]  (1)

He reduced the sampling interval from \( T \) to 0.5 \( T \). Reducing the sampling interval to 0.5 \( T \) modifies the transfer function of the Simpson integrator to

\[ H_{0.5S}(z) = \frac{T}{6} \frac{1 + 4z^{-1/2} + z^{-1}}{1 - z^{-1}} \]  (2)

Thus, the relationship between the transfer functions of the Simpson and the modified Simpson integrators may be expressed as

\[ H_{0.5S}(z) = \frac{1}{2} H_S(z^{1/2}) \]  (3)

The problem of how to implement the half sample delay, or for that matter any fractional delay, was developed with the advances in multirate digital signal processing as implementation of successive interpolation and decimation and vice versa. Finding efficient design methods of fractional delay filters is an active research area [4–16].

To implement the fractional delay element \( z^{-1/2} \), Tseng employed two of the methods proposed by Laakso et al. [9], the finite impulse response (FIR) fractional delay filter and the infinite impulse response (IIR) fractional delay filter. Thus, a pure integer delay \( z^{-N} \) is cascaded with the modified Simpson integrator as

\[ G_N(z) = z^{-N} H_{0.5S}(z) \]

\[ = \frac{T}{6} \frac{z^{-N} + 4z^{-(N+1/2)} + z^{-(N-1)}}{1 - z^{-1}} \]  (4)

The problem is reduced to designing the fractional delay element \( z^{-D} \), where \( D = N + (1/2) \).

Tseng found that the integrators obtained using the FIR-based design have better performance than those obtained using the allpass-based design; thus, only his results using the FIR-based design will be reported in
Fig. 1  Ideal differentiator (reproduced from [1])

a Magnitude response of an ideal differentiator
b Phase response of an ideal differentiator
c Magnitude response of an ideal half sample delay
d Phase response of an ideal half sample delay
e Magnitude response of an ideal differentiator delayed by half a sample
f Phase response of an ideal differentiator delayed by half a sample
For the FIR-based design, he reported two cases. In the first case, he chose \( D = \frac{1}{2} \) and \( N = 0 \), which yielded the half sample delay approximation \( z^{-1/2} \approx \frac{1}{2} + (1/2)z^{-1} \). The resulting transfer function is the same as that of the conventional trapezoidal rule. For the second case, he chose \( D = 1.5 \) and \( N = 1 \), which yielded the one and a half sample delay approximation \( z^{-3/2} \approx -1/8 + (3/4)z^{-1} + (3/8)z^{-2} \). The resulting transfer function of the integrator is

\[
G_1(z) = \frac{T}{12} \left( -1 + 8z^{-1} + 5z^{-2} \right) \quad (5)
\]

The magnitude of the frequency response of the above integrator is better than that of the trapezoidal integrator.
However, it introduces an additional sample delay, and it loses the $-90\degree$ phase shift.

This paper addresses the use of fractional delay methods to control the magnitudes and/or phases of integrators and differentiators with the aim of bringing them to approximate the magnitudes and/or phases of ideal integrators and differentiators. Applications to the first-order, Al-Alaoui integrator and differentiator are presented.

The magnitude and phase responses of the Al-Alaoui integrator are shown in Fig. 2. The Al-Alaoui integrator is obtained by interpolating the transfer function of the trapezoidal integrator with either the forward rectangular or the backward rectangular integrators [17–19]. The transfer functions of the forward rectangular, the backward rectangular and the trapezoidal digital integrators are given, respectively, by

$$H_{\text{FR}}(z) = T \left( \frac{z^{-1}}{1 - z^{-1}} \right)$$  \hspace{1cm} (6a)

$$H_{\text{BR}}(z) = T \left( \frac{1}{1 - z^{-1}} \right)$$  \hspace{1cm} (6b)

$$H_{\text{TR}}(z) = \frac{T}{2} \left( \frac{1 + z^{-1}}{1 - z^{-1}} \right)$$  \hspace{1cm} (7)

where $T$ is the sampling period. The interpolation takes the form

$$H_{\text{AI}}(z) = aH_{\text{BR}}(z) + (1 - \alpha)H_{\text{TR}}(z)$$  \hspace{1cm} (8)

Substituting (6b) and (7) into (8), with $\alpha = 3/4$ yields the minimum phase integrator

$$H_{\text{AI}}(z) = 7 \frac{T}{8} \left( \frac{z + (1/7)}{z - 1} \right) = 7 \frac{T}{8} \left( \frac{1 + (1/7)z^{-1}}{1 - z^{-1}} \right)$$  \hspace{1cm} (9)

It is to be noted that a maximum phase integrator would have been obtained if the forward rectangular rule, (6a), was used instead of (6b) in (8). However, reflecting the resulting zero outside the unit circle to inside the unit circle and compensating for the magnitude yields the minimum phase integrator of (9) [18–20]. We can see that the phase response is almost linear.

Inverting the transfer function of (9) results in the transfer function of the differentiator [4, 7], known as the Al-Alaoui

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**Fig. 4** Phase response of the delayed Al-Alaoui integrator

**Fig. 5** Delayed Al-Alaoui integrator: FIR filter case (Lagrange interpolation)

*a* Absolute magnitude error with respect to the ideal response  
*b* Phase response

**Fig. 6** Delayed Al-Alaoui integrator: allpass filter case

*a* Absolute magnitude error with respect to the ideal response  
*b* Phase response
operator [20, 21], and is given as

\[ H_{AD}(z) = \frac{8(z - 1)}{7T(z + (1/7))} = \frac{8(1 - z^{-1})}{7T(1 + (1/7)z^{-1})} \]  

(10)

The resulting differentiator is stable, with a pole at \( z = -1/7 \), and has an almost linear phase as shown in Fig. 3c.

A fractional delay is added to the integrator leading to an almost constant phase response of \(-90^\circ\).

We would like to reach an almost constant \(90^\circ\) phase for the Al-Alaoui differentiator. Hence, the reverse of the process mentioned previously for linearising the phase of the ideal differentiator is applied: introducing a half sample advance to the Al-Alaoui differentiator will lead to an almost constant \(90^\circ\) phase response.

It should be noted that the approach applies to other differentiators and integrators with linear phases, or approximately linear phases, such as the second-order Al-Alaoui integrators and differentiators [22–24]. In fact, it was applied in [3] to the Simpson integrator, which becomes one of the second-order Al-Alaoui differentiators when its transfer function is inverted. However, this paper will delinate only the first-order Al-Alaoui integrators and differentiators and, for brevity, the term ‘first-order’ will be suppressed.

2 Conventional Al-Alaoui integrators delayed by half a sample

2.1 Ideal fractional delay

The transfer function of the Al-Alaoui integrator is given by (9). Its phase response shown in Fig. 2c is almost linear. However, our objective is to obtain a constant phase response of \(-90^\circ\). Delaying the response of the conventional Al-Alaoui integrator by half a sample yields

\[ H_2(z) = z^{-1/2}H_{AI}(z) = \frac{T}{8}z^{-1/2}\left(\frac{1 + (1/7)z^{-1}}{1 - z^{-1}}\right) \]  

(11)

The magnitude response of this integrator is the same as that of the conventional integrator shown in Fig. 2a and b. We can clearly see from Fig. 4 that the phase is almost constant at \(-90^\circ\), with less than 10\% deviation, ranging from \(-98^\circ\) to \(-90^\circ\) approximately. The maximum error is approximately 8.213\% (i.e. 9.1\%) at 0.5447 of the Nyquist frequency.

An ideal half sample delay is available in applications where multirate processing is involved. However, in other cases, we need to approximate the value between samples to obtain the required fractional delay. For this purpose, we use two methods among the tools discussed in [9]: the
FIR case (Lagrange interpolation) and the allpass filter case. These methods are attractive because the delay filter coefficients are obtained in closed form.

2.2 Fractional delay approximated by an FIR filter (Lagrange interpolation)

With this approximation, a total delay $D$ is approximated by

$$z^{-D} \simeq \sum_{n=0}^{L} h(n) z^{-n}, \quad h(n) = \prod_{k=0, k \neq n}^{L} \frac{D-k}{n-k}$$

(12)

The filter-order $L$ is chosen such that $(L - 1)/2 \leq D \leq (L + 1)/2$.

Moreover, the total delay $D$ is expressed as $D = N + d$, where $N$ is the integer part of the delay and $d$ the fractional part. The integer part of the delay is needed to increase the filter order to obtain a better approximation of the fractional delay. However, this will affect the phase response and the required $-90^\circ$ phase will not be obtained. To solve this problem and delay the response only by the fractional part $d$, we multiply the integrator’s transfer function by $z^{-N-d}/z^{-N}$ instead of $z^{-N-d}$, where $z^{-N-d}$ is approximated by the FIR filter in (12).

The magnitude and frequency responses of the delayed Al-Alaoui integrator, with the delay approximated by

![Fig. 9 Double-rate Al-Alaoui integrator: allpass filter case](image)

- Absolute magnitude error with respect to the ideal response
- Phase response

Fig. 10 Phase response of the combined double-rate Al-Alaoui integrator

Lagrange interpolation with $N = 5$, $L = 10$ and $d = 0.5$ are shown in Fig. 5. The magnitude error is the same as the conventional Al-Alaoui integrator at low frequencies. However, it increases after 0.45 of the Nyquist frequency. The phase is almost $90^\circ$ at low frequencies. The maximum phase error is $7.83^\circ$ at 0.488 in the interval $[0 \text{ } 0.8]$ of the Nyquist frequency. The interval $[0 \text{ } 0.8]$ is the interval where the error is tolerable for the Lagrange interpolation case. The inaccuracies at high frequencies are because of the approximation error.

2.3 Fractional delay approximated by an allpass filter (Thiran allpass IIR filter)

With this approximation, a total delay $D$ is approximated by

$$z^{-D} \simeq \frac{a_N + a_{N-1}z^{-1} + \cdots + a_1z^{-(N-1)} + z^{-N}}{1 + a_1z^{-1} + \cdots + a_{N-1}z^{-(N-1)} + a_Nz^{-N}},$$

(13)

$$a_k = (-1)^k C^N_k \prod_{n=0}^{N} \frac{N-n-D}{N-k-n-D}$$

where $C^N_k = N!/k!(N-k)!$ and $D = N + d$. To delay the response only by the fractional part $d$, we multiply the
integrator’s transfer function by $z^{-2N-d}/z^{-N}$ instead of $z^{-N-d}$, where $z^{-N-d}$ is approximated by the allpass filter in (13). The frequency responses of the delayed Al-Alaoui integrator, with the delay approximated by the Thiran allpass IIR filter with $N = 5$ and $d = 0.5$ are shown in Fig. 6.

The magnitude error is the same as the conventional Al-Alaoui. Hence, using the allpass filter approximation to enhance the phase response did not entail any penalty on the magnitude response. The phase is almost $90^\circ$ at low frequencies. The maximum phase error is $7.27^\circ$ at 0.43 in the interval $[0, 0.675]$ of the Nyquist frequency. The interval $[0, 0.675]$ is the interval where the error is tolerable for the allpass filter case. The inaccuracies at high frequencies are, again, because of the approximation error.

3 Double-rate Al-Alaoui integrator

3.1 Ideal fractional delay

Using a sampling period of $T/2$ (i.e. doubling the sampling rate), (9) becomes

$$H_3(z) = \frac{7}{8} \left( \frac{1 + (1/7)z^{-1}}{1 - z^{-1}} \right)$$

$$= \frac{7}{16} \left( \frac{1 + (1/7)z^{-1}}{1 - z^{-1}} \right)$$

Multiplying the numerator and denominator of (9) by $(1 + z^{-1/2})$ yields

$$H_3(z) = \frac{7}{16} \left( \frac{T}{16} \right) \frac{1 + (8/7)z^{-1/2} + (1/7)z^{-1}}{1 - z^{-1}}$$

$$= \frac{T}{16} \left( \frac{7 + 8z^{-1/2} + z^{-1}}{1 - z^{-1}} \right)$$

The magnitude error of the double-rate Al-Alaoui integrator at high frequencies is superior to that of the double-rate Simpson integrator as shown in Fig. 7a. The phase response is shown in Fig. 7b. It is to be noted that the double-rate operation does not change the phase.

We implement the fractional delay element using the tools in [9] by cascading a pure integer delay $z^{-N}$ with the proposed integrator. However, to keep the almost linear phase of the conventional Al-Alaoui integrator, we also divide the transfer function by $z^{-N}$. Hence

$$H_3(z) = \frac{z^{-N}H_3(z)}{z^{-N}}$$

$$= \frac{T}{16} \left( \frac{7z^{-N} + 8z^{-(N+1/2)} + z^{-(N+1)}}{z^{-N} - z^{-(N+1)}} \right)$$

Again the problem is to approximate the delay element $z^{-(N+1/2)}$. Note that when multirate processing is
available, the values half way between the samples are already available and there is no need for approximation or use of $z^{-N}$. (The values of Fig. 7 are then realisable in practice.) We use the two methods we employed in Section 2: The FIR filter method using Lagrange interpolation and the allpass filter method.

### 3.2 Double-rate Al-Alaoui integrator with fractional delay approximated by an FIR filter (Lagrange interpolation)

We implement (16) using the fractional delay approximation of (12). The results are shown in Fig. 8. By comparing Fig. 8a with Fig. 2b, we see that the magnitude error at low frequencies is lower for the FIR case. However, it increases slightly above that of the conventional Al-Alaoui integrator at high frequencies. The phase response is linear in the low-frequency range (Fig. 8b) similar to Fig. 7b, but linearity is lost at high frequencies because of the approximation filter error.

### 3.3 Double-rate Al-Alaoui integrator with fractional delay approximated by an allpass filter

We implement (16) using the fractional delay approximation of (13). The results are shown in Fig. 9. By comparing Fig. 9a with Fig. 2b, we see that the magnitude error at low frequencies is lower for the allpass case. However, it increases slightly above that of the conventional Al-Alaoui integrator at high frequencies. The phase response is linear in the low-frequency range (Fig. 9b) similar to Fig. 7b, but linearity is lost at high frequencies because of the approximation filter error.

### 3.4 Combined double-rate Al-Alaoui integrator (combination of FIR and allpass cases)

In Fig. 8a, the response of the approximated filter is a little below the ideal response at high frequencies. However, in Fig. 9a, the response of the approximated filter is a little above the ideal response at high frequencies. This leads us to study the possibility of combining the two filters in order to improve the magnitude response (minimise the magnitude error with respect to the ideal case). We seek a result of the form

$$H_{\text{combined}} = aH_{\text{FIR}} + bH_{\text{allpass}}$$

(17)

where

$$a = \frac{\int_0^\pi |H_{\text{allpass}}| - 1/\omega|d\omega}{\int_0^\pi |H_{\text{allpass}}| - 1/\omega|d\omega + \int_0^\pi |H_{\text{FIR}}| - 1/\omega|d\omega}$$
We found $a = 0.55$ and $b = 0.45$. Hence, we define

$$H_3_{\text{combined}} = 0.55 H_3_{\text{FIR}} + 0.45 H_3_{\text{allpass}}$$

The magnitude errors of the combined double-rate Al-Alaoui integrator is almost identical to that of the ideal double-rate Al-Alaoui integrator (Fig. 7a), and is omitted for brevity. The phase response is shown in Fig. 10.

## 4 Double-rate Al-Alaoui integrator delayed by 0.2 samples

### 4.1 Ideal fractional delay

In Section 2, we tried to enhance the phase response of the Al-Alaoui integrator to obtain a constant phase of $-90^\circ$. In Section 3, we doubled the sampling rate to improve the magnitude response of the integrator. Now, we would like to combine the results to improve both the magnitude and phase responses at the same time. We want the double-rate Al-Alaoui integrator to yield a constant phase response of $-90^\circ$.

From Fig. 7b, we can see that the phase response is almost linear. Hence, adding the proper delay would give the desired result. The phase response at the normalised Nyquist frequency in Fig. 7b is $-53.152^\circ$. The required delay $d$ should be such that $-53.152 + 180 d = -90$, from which we obtain $d = -0.2047$. We need to add a delay of 0.2047 samples to the transfer function of (15)

$$H_4(z) = z^{-0.2047} H_1(z)$$

In the case when such a rate can be obtained (multirate applications), the magnitude error is the same as in Fig. 7a, and the phase response is shown in Fig. 11, where we can clearly see that the phase is almost constant at $-90^\circ$. Note that the range extends approximately from $-91.2^\circ$ to $-90^\circ$. The maximum error is around 1.2$^\circ$ (i.e. 1.3%) at 0.577 of the Nyquist frequency. This is indeed a better result than that of Fig. 4. (Using a delay of 0.2 samples instead of 0.2047 leads to a maximum error of 0.865$^\circ$ at the Nyquist frequency. Hence, not sticking to the exact value does not have negative consequences in this case.)
Now if we want to approximate the fractional delay by the methods used previously, we need to implement

\[ H_3(z) = \frac{z^{-N} - 0.2047}{z^{-2N} - z^{-2N+1}} \]

(21)

In fact, we need to multiply \((z^{-N}/z^{-N})H_1(z)\), implemented as discussed in Sections 2.2 and 2.3, by the approximation of \(z^{-N} - 0.2047/z^{-N}\) using any of the two approximation methods.

4.2 Delayed double-rate Al-Alaoui integrators with fractional delay approximated by an FIR filter (Lagrange interpolation)

Here, both the half sample and 0.2 sample delays are approximated using the Lagrange interpolation method. The results of approximating all the delays in the transfer function of (21) by Lagrange interpolation are shown in Fig. 12.

By comparing Fig. 12a with Fig. 2b, we see that the magnitude error at low frequencies is lower for the FIR case. However, it increases slightly above that of the conventional Al-Alaoui integrator at high frequencies. The phase response is almost constant in the low-frequency range (Fig. 12b) similar to Fig. 11, but this is not the case at high frequencies because of the approximation filter error.

4.3 Delayed double-rate Al-Alaoui integrators with fractional delay approximated by an allpass filter

Here, both the half sample and 0.2 sample delays are approximated using the allpass filter method (Thiran IIR). The results of approximating all the delays in the transfer function of (21) by the allpass filter method are shown in Fig. 13.

By comparing Fig. 13a with Fig. 2b, we see that the magnitude error at low frequencies is lower for the allpass case. However, it increases slightly above that of the conventional Al-Alaoui integrator at high frequencies. The phase response is almost constant in the low-frequency range (Fig. 13b) similar to Fig. 11, but this is not the case at high frequencies because of the approximation filter error.
4.4 Combined delayed double-rate Al-Alaoui integrators (combination of FIR and allpass cases)

In Fig. 12a, the response of the approximated filter is a little below the ideal response at high frequencies. However, in Fig. 13a, the response of the approximated filter is a little above the ideal response at high frequencies. Hence, we combine the two filters in order to improve the magnitude response (minimise the magnitude error with respect to the ideal case), as in Section 3.4. Applying (13), we found $a = 0.45$ and $b = 0.55$

$$H_{\text{combined}} = 0.45H_{\text{FIR}} + 0.55H_{\text{allpass}}$$  \hspace{1cm} (22)

A comparison between the magnitude errors of the combined delayed double-rate Al-Alaoui, ideal double-rate Al-Alaoui and conventional Al-Alaoui integrators shows the superiority of the first (Fig. 14a). The phase response is shown in Fig. 14b.

5 Applying the same approach to the Al-Alaoui differentiator

The transfer function of the Al-Alaoui differentiator is given by (10). The magnitude and phase responses of this differentiator are shown in Fig. 3. We can see that the phase response is almost linear. To get a constant phase response of 90° and to improve the magnitude response of this differentiator, we use the same approach used for the integrator. Each of the transfer functions that should be considered corresponds to the inverse of a transfer function used in one of the previous sections. The main problem to be considered is the use of a fractional phase advancement instead of a fractional delay. A filter with a response $z^{-1/2}$ is non-causal. In practice, a response of $\frac{z^{-N}}{z^{-N}-1/2}$ is used, where $N$ is an integer delay. Hence, we need to delay the denominator by half a sample when we need a phase advance of half a sample. The problem is again the fractional delay problem, which can be solved using any of the filters presented in Sections 2.2 and 2.3. Some of the results, for brevity, are shown in Figs. 15–17.

6 Reducing the approximation error

This section discusses some solutions to improve the performance of the delay approximation filters. Fig. 18a shows the group delay of the FIR filter approximating half a sample delay with $N = 5$ and $L = 10$. This is the filter used in all the previously discussed simulations involving Lagrange interpolation. The result of increasing the filter order form 10 to 80 is shown in Fig. 18b ($N = 40$, $L = 80$). The result of applying this filter instead of the previous one to the delayed double-rate Al-Alaoui integrator is presented in Fig. 18c, and to the advanced double-rate Al-Alaoui differentiator is presented in Fig. 18d. Comparison of Fig. 18c and d with Figs. 12b and 16b, respectively, definitely proves the better performance of the filter in Fig. 18b. The same reasoning applies to the allpass filter case (compare Figs. 19b with 13b and Figs. 19c with 17b).

The improvements mentioned above come at the expense of a great increase in filter complexity. We said that to approximate $z^{-0.5}$, we approximate $z^{-N-0.5}$ and divide by $z^{-N}$. But $z^{-N}$ has a purely linear phase. The error comes from $z^{-N-0.5}$. Hence, dividing by an approximating function having a similar phase error might lead to phase errors cancelling out each other: for example, $z^{-0.5} = z^{-N-0.6}/z^{-N-0.1}$ (or $z^{+0.5} = z^{-N-0.1}/z^{-N-0.6}$ for the differentiator case). We can approximate $z^{-N-0.6}$ and $z^{-N-0.1}$ using two separate simple FIR filters with $N = 5$ and $L = 10$ [the parameters are computed using (12)]. The division will lead to major amelioration regarding phase error. In fact, we can see this in Fig. 20 where we used the previous method to compute the phase response.


Fig. 19 Improving the performance of the allpass filter

a Group delay of an allpass fractional delay filter with $N = 40$ and $d = 0.5$

b Phase response of delayed double rate Al-Alaoui integrator using allpass filter from $a$

c Phase response of advanced double rate Al-Alaoui differentiator using allpass filter from $a$
of the delayed conventional Al-Alaoui integrator. Using the method to compute the phase response of the advanced conventional Al-Alaoui differentiator (the FIR filter used approximates $z^{-0.5}$ by approximating the ratio $z^{-N-0.1}/z^{-N-0.1}$), yields a phase response which is simply minus the phase response of the integrator and the corresponding figure is omitted for brevity. Comparison of Fig. 20 with Fig. 5b, and the minus of Fig. 20 with Fig. 15b, respectively, shows the usefulness of this method.

We should also note that if sufficient oversampling is possible, we can move the Nyquist frequency to less than half of the sampling frequency where the error is small.

To demonstrate the efficiency of the proposed design method, we compare it to the conventional least-squares (LS) approach in the frequency domain. Hence, we use the Matlab function ‘invfreqz’: $[b, a] = \text{invfreqz}(h, w, n, m)$ which returns the real numerator and denominator coefficients in vectors $b$ and $a$ of the transfer function whose complex frequency response is given in vector $h$ at the frequency points specified in vector $w$. Scalars $n$ and $m$ specify the desired orders of the numerator and denominator polynomials

$$H(z) = \frac{B(z)}{A(z)} = \frac{b(1) + b(2)z^{-1} + b(3)z^{-2} + \cdots + b(n+1)z^{-n}}{a(1) + a(2)z^{-1} + a(3)z^{-2} + \cdots + a(m+1)z^{-m}}$$

(23)

‘invfreqz’ finds $b$ and $a$ in

$$\min_{b,a} \sum_{k=1}^{p} W(k)|h(k)M(\omega(k)) - B(\omega(k))|^2$$

(24)

by creating a system of linear equations and solving them. Here $A(\omega(k)) = A(z)|_{z=\omega(k)}$, $B(\omega(k)) = B(z)|_{z=\omega(k)}$, $p$ is the number of frequency points (the length of $h$ and $\omega$), and $W(k)$ is a weighting vector. This algorithm solves the direct problem of minimising the weighted sum of the squared error between the actual and the desired frequency response points.

We consider, as an example, the case of the conventional Al-Alaoui integrator delayed by half a sample with $N = 5$, and compare it to a filter of the same order approximating the ideal integrator, obtained using (23) and (24). The results displayed in Fig. 21 show the superiority of the proposed design method over the LS method. The results were obtained using uniform weighting, $W(k) = 1$ for all $k$, while $\omega(k = 1) = 0$ and $\omega(k = p) = \pi$ in steps of 0.01.

7 Conclusions

A fractional delay was added to the conventional first-order Al-Alaoui integrators, and an almost constant phase of $-90^\circ$ was obtained. Then the sampling rate was doubled by interpolating the values between the samples, and the magnitude error was reduced. Delaying the double-rate integrator led to improvements in both magnitude and phase responses. In addition to the ideal delay (multirate applications), approximations using the FIR and allpass filter were investigated. The allpass filter yielded better results than the FIR filter with respect to the magnitude error. A proper combination of both approximations usually leads to the best results. The coefficients of each filter in the combination were chosen based on the observation of the filters’ responses.

Similar procedures were applied to the conventional first-order Al-Alaoui differentiators using a fractional advance, instead of a fractional delay, and an almost constant phase of $+90^\circ$ was obtained.

The approach may be applied to other differentiators and integrators with linear, or approximately linear, phases.

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