Low-Frequency Differentiators and Integrators for
Biomedical and Seismic Signals

Mohamad Adnan Al-Alaoui

Abstract—A general active-network synthesis approach to inverse
system design is introduced. The approach is applied to a passive RC
differentiator and a passive RC integrator to obtain, respectively, a very
low-frequency differential integrator and a very low-frequency differential
differentiator. The frequency ranges of the proposed circuits, from dc to
a few hundred hertz, are particularly suitable to the frequency ranges of
biomedical and seismic signals. The advantages of the proposed circuits
are delineated and include single time constants, dc stable integrators, and
resistive input differentiators. Noninverting and inverting differentiators
and integrators could be obtained by grounding one of the input terminals
in the differential configurations.

Index Terms—Active-network synthesis, biomedical signals, differentia-

tors, integrators, inverse system design, low frequency, seismic signals.

I. INTRODUCTION

This paper introduces a differential integrator and a differential differ-

etiator for the very low-frequency signals, from dc to a few hun-

dred hertz, that arise in the measurements of biomedical signals [1] and

seismic signals [2]. The resulting low-frequency integrators and differ-

etiators are eminently suitable for use with biomedical and seismic signals

where the conventional wisdom advises us to seek DSP solutions for such low frequencies, because they are deemed to be not prac-

tically attainable by using analog filters [2]. The analog filters are often

simpler to implement and do not require the possible added complexity

and cost of analog-to-digital and digital-to-analog converters.

The basic concept of both circuits is to employ an inverse system
design to obtain the inverse of a passive RC integrator and a passive RC
differentiator which yield, respectively, active RC differentiator, and

integrator [3]–[6]. The traditional differential integrator and differential
differentiator are shown in Figs. 1 and 2, respectively [7].

The proposed circuits have the following advantages over the tradi-
tional circuits.

1) Single time constants are obtained for both circuits.

2) Resistive inputs, without using input buffers, are obtained for

both circuits.

3) The integrator is dc stable and the differentiator action ceases at

high frequencies.

4) Reasonably high-quality integration and differentiation can be

obtained while avoiding the dc instability for the integrator and

the instability caused by the capacitive input of the differentiator.

5) It is easier to control the common-mode rejection ratio (CMRR)

with a single time constant, by choosing appropriately small tol-

erance resistances and an appropriate amplifier.

6) Functional-bandpass integrators and differentiators are obtained

with control over both frequency limits of the bandpass. The lim-

ited bandwidth mitigates the noise contribution.

II. THE BASIC CONCEPT: ACTIVE NETWORK SYNTHESIS OF INVERSE

SYSTEM DESIGN

The basic concept came from observing that the inverse of a passive
RC differentiator approximates the ideal integrator. Also the inverse of

a passive RC integrator approximates an ideal differentiator.

The approach to inverse-system synthesis is shown in Fig. 3. The

transfer function relating the output voltage to the input voltage of

Fig. 3 is [3]–[5]

\[
\frac{V_o(s)}{V_i(s)} = \frac{k}{1 + kH(s)}.
\]  

(1)
The system will approximate \(1/H(s)\) in the frequency range where

\[
k H(s) \gg 1. \tag{2}
\]

The proposed approach to active-network synthesis, with dual-input configuration, is shown in Fig. 4. The open-loop transfer function of the operational amplifier used in the differential stage will be denoted as \(A(s)\) while the one used in the buffer stage, implemented as an isolation amplifier, will be designated as \(A_H(s)\). The transfer function relating the output voltage of the isolation-stage operational amplifier to its input can be shown to be \(1/[1/A_H(s)] + 1 \approx 1\) for low frequencies. It should be noted that the isolation amplifier can be dispensed with if \(r\) is chosen large enough such that the loading of the circuit composed of \(Z_1\) and \(Z_2\) is negligible. The derivation of the transfer function of the differential circuit of Fig. 4, relating its output voltage to its input voltage, is outlined in the following.

Let \(V^{-}, V^{+}\), and \(V_X\) designate the voltages in Fig. 4, with respect to ground, from the inverting terminal of the integrator stage operational amplifier, the noninverting terminal of the operational amplifier to ground and the voltage across the \(Z_1\) impedance, respectively.

Nodal analysis, similar to those which were carried out in [3]–[6], assuming that the output voltage of the isolation amplifier has the value \(V_X\) and that for the low-frequency range \(A(s)\) is very large, \(|A(s)| \gg k\), and \(k \gg 1\), yields

\[
\frac{V_o}{V_1 - V_2} \approx \frac{k(Z_1 + Z_2)}{kZ_1 + Z_2}. \tag{3}
\]

In the limit if \(k\) is infinite, which corresponds to removing the feedback resistor labeled \(kr\) in Fig. 5, \(3\) simplifies further to

\[
\frac{V_o}{V_1 - V_2} \approx \frac{Z_1 + Z_2}{Z_1}. \tag{4}
\]

The system might still be stable, since there remains a feedback path through the isolation amplifier. The stability will depend on \(Z_1\) and \(Z_2\) and should be investigated for each case separately.

III. THE PROPOSED LOW-FREQUENCY DIFFERENTIAL INTEGRATOR

A. The Derivation of the Transfer Function

The derivation of the transfer function of the proposed differential integrator of Fig. 5, relating its output voltage to its input voltage, is outlined in the following.

Fig. 5 is obtained from Fig. 4 with \(Z_1 = R\) and \(Z_2 = 1/C s\). Hence \(3\) yields

\[
\frac{V_o}{V_1 - V_2} \approx \frac{k(1 + RC/s)}{1 + kRC/s}. \tag{5}
\]

If \(RC/s \ll 1\), or equivalently \(\omega \ll 1/(RC)\), \(5\) simplifies to

\[
\frac{V_o}{V_1 - V_2} \approx \frac{k}{1 + kRC/s}. \tag{6}
\]

Equation \(6\) represents an integrator for the frequencies in the range

\[
\frac{1}{kRC} \ll \omega \ll \frac{1}{RC}. \tag{7}
\]

The circuit acts as an amplifier for dc inputs. The dc gain is obtained from \(6\) by substituting zero for \(s\) to obtain the dc value of \(k\). Thus, the dc gain may be increased or decreased as desired by respectively increasing or decreasing \(k\). Note that the resulting dc stability is one of the advantages of the circuit over the traditional Miller integrator, which is dc unstable due to the high gain, \(A_v\), of the operational amplifier. Typically \(A_v \approx 10^5\) for LM 741 operational amplifier.

B. The Quality Factor \(Q\) of the Differential Integrator

If we express the transfer function of an integrator as \(3\)–\(5\)

\[
T(j\omega) = \frac{1}{R(\omega) + jX(\omega)} \tag{8}
\]

then, the \(Q\)-factor of the integrator is defined as

\[
Q = \frac{X(\omega)}{R(\omega)}. \tag{9}
\]

The traditional differential integrator, with a \(Q\) value equal to that of the Miller integrator, has \(Q = -|A| = -|\omega_o/\gamma| = -|\omega_o/\omega|\).

For the ideal or low-frequency case, we obtain, from \(6\), the value

\[
Q = kRC\omega \gg 1. \tag{10}
\]

Thus, for \(RC\omega = 0.1\) and \(k = 1000\), we obtain a \(Q\) value of 100.

C. The Case of Infinite \(k\)

Note that \(6\) represents a stable system, with its pole in the left half of the \(s\)-plane. In the limit, if \(k\) is infinite, the right hand side of \(6\) will simplify to \(1/RC\) which has a pole at the origin and the system will not be stable. However, if the capacitance is shunted with a resistor, the
resulting system will be stable and will still act as an integrator. In this case, let \( Z_1 = R_1 \) and \( Z_2 = R_2/(1 + R_2 C s) \). Thus, we will have
\[
\frac{V_o}{V_1 - V_2} \approx \frac{(R_1 + R_2)(1 + R_1 R_2)}{R_1 (1 + R_2 C s)}.
\]
Equation (11) represents a stable and a minimum phase system, with a single pole and a single zero in the left half of the s plane. It can be verified that the above circuit acts as an integrator in the frequency range
\[
\frac{1}{R_2 C} \ll \omega \ll \frac{1}{R_2 C} + \frac{1}{R_1 C}.
\]
For brevity, this case will not be further elaborated.

**IV. PROPOSED LOW-FREQUENCY DIFFERENTIAL DIFFERENTIATOR**

**A. The Derivation of the Transfer Function**

The derivation of the transfer function of the differential differentiator of Fig. 6, relating its output voltage to its input voltage, is outlined in the following.

In this case, Fig. 6 is obtained from Fig. 4 with \( Z_1 = 1/C s \) and \( Z_2 = R \), hence (3) yields
\[
\frac{V_o}{V_1 - V_2} \approx \frac{k(1 + RC' s)}{k + RC s}.
\]
Equation (13) represents a differentiator for frequencies in the range
\[
\frac{1}{RC} \ll \omega \ll \frac{k}{RC}.
\]
Indeed in the frequency range specified by (14), (13) can be approximated as
\[
\frac{V_o}{V_1 - V_2} \approx RC' s.
\]
The circuit acts as an amplifier for dc inputs. The dc gain is obtained from (13) by substituting zero for \( s \) to obtain a dc gain of \( \approx 1 \).

**B. The Quality Factor \( Q \) of the Differential Differentiator**

If we express the transfer function of a differentiator as [6]
\[
T(j\omega) = R(\omega) + jX(\omega)
\]
then the \( Q \)-factor of the differentiator is defined as
\[
Q = \frac{X(\omega)}{R(\omega)}.
\]
The traditional differential differentiator, with a \( Q \) value equal to that of the Miller integrator, has \( Q = -|A| = -|\omega_1/s| = -|\omega_1/\omega| \).

For the ideal or low-frequency case, we obtain, from (13), with \( RC \gg 1 \), the value
\[
Q = k/RC\omega.
\]
Thus for \( RC\omega = 1 \) and \( k = 100 \), we obtain a \( Q \) value of 100.

**C. The Case of Infinite \( K \)**

Note that for an infinite \( k \) the right hand side of (13) reduces to \( 1/(1 + RC s) \) and the system is still a stable system. Thus the feedback resistor could be removed and the range of frequency is \( \omega \gg 1/(RC) \).

The differentiator is no longer confined to the low-frequency range. The high frequency range of the differentiator is limited by the gain-bandwidth characteristics of the operational amplifier. For brevity, this case will not be further elaborated.

**V. SIMULATION AND EXPERIMENTAL RESULTS**

**A. Obtaining Two Voltages of Opposite Polarities and Equal Magnitudes**

The circuit of Fig. 7 was used in all the simulation and experimental work. Voltages of opposite polarities, and with roughly equal delay and magnitude, are obtained from the outputs of the two LM741 operational amplifiers in Fig. 7, upon the application of a voltage at the input terminals designated as \( V \). One of the operational amplifiers is configured as an inverting amplifier with a gain of \(-1\), while the other is configured as a unity gain voltage-follower amplifier. The output of the voltage follower was connected to the inputs designated as \( V_2 \) in Figs. 5 and 6, while the output of the inverting amplifier was connected to the inputs designated as \( V_1 \) in Figs. 5 and 6.

**B. The Simulation Results**

The differential integrator circuit of Fig. 5 was simulated by using PSpice with \( r = 100 \) k\( \Omega \), \( k = 1000 \), \( C = 100 \) \( \mu \)F, \( R = 100 \) k\( \Omega \) and LM741 for the operational amplifier with dc bias of \( \pm 15 \) V. The simulation results for the magnitude and phase are shown in Fig. 8. The simulation shows that, with the above values, the circuit acts as an integrator for the low-frequency range in the neighborhood of 1 mHz, in agreement with (7). Different frequency ranges may be obtained by varying \( R \) and/or \( C \) appropriately.
The differential differentiator circuit of Fig. 6 was simulated by using PSPICE with $r = 100 \, k\Omega$, $k = 1000$, $C = 100 \, \mu\text{F}$, $R = 100 \, K\Omega$ and LM741 for the operational amplifier with dc bias of $\pm 15 \, \text{V}$. The simulation results for the magnitude and phase are shown in Fig. 9. The simulation shows that, with the above values, the circuit acts as a differentiator for the low-frequency range in the neighborhood of $1 \, \text{Hz}$, in agreement with (14). Different frequency ranges may be obtained by varying $R$ and/or $C$ appropriately.

C. The Experimental Results

The experimental set up utilized, in addition to the circuit of Fig. 7 and the circuit being tested corresponding to Fig. 5 or Fig. 6, a power
supply to provide the dc bias of ±15 V, function generator HP33120 which provided rectangular wave voltages to the inputs of the integrator circuit and triangular wave voltages to the input of the differentiator circuit, and signal analyzer HP89410A which provided the displays of the input and output waveforms shown in Figs. 10 and 11.

Fig. 10 shows the experimental results of the differential integrator of Fig. 5, with $r = 100 k\Omega$, $k = 1000$, $C = 10 \mu F$, $R = 560 \Omega$ and LM741 for the operational amplifier with dc bias of ±15 V. The upper trace shows the input square waveform with a frequency of 340 mHz. The bottom trace shows the resulting triangular waveform at the output of the operational amplifier. Thus, a good integration action is obtained by using the proposed circuit. Note that in Fig. 5 the addition of resistor of value $R = 560 \Omega$ between the negative input terminal of the isolation operational amplifier and ground was necessary to ameliorate the offset voltage at the output. Keeping the same value of the resistance seen from each of the two input terminals of the operational amplifiers to ground reduces the offset voltage at the output since the values of the dc currents into the two terminals are almost equal.

Fig. 11 shows the experimental results of the differential differentiator of Fig. 6, with $r = 100 k\Omega$, $k = 1000$, $C = 330 \mu F$, $R = 100 \Omega$.
A stable inverting integrator with an extended high-frequency range, A novel differential differentiator, A differential integrator with a built-in high frequency compensator. Each employs a single time constant, has a resistive input, and is obtained by using the proposed circuit. It was not necessary to add a resistor between the negative input terminal of the isolation operational amplifier and ground since a dc input to a differentiator produces a zero output voltage.

D. Comparison Among the Theoretical, Simulation, and Experimental Results

The simulation and experimental results verify the predicted frequency ranges of (7) for the integrator and (14) for the differentiator. Note also that the clean waveforms of Figs. 10 and 11 indicate high signal to noise ratios for both circuits.

It should be pointed out that the case of infinite $k_r$ with the feedback resistor $k_r$ removed, produced good experimental and simulation results which were omitted for brevity.

VI. Conclusion

An active-network synthesis of inverse system design is presented. The synthesis is general and can be applied with different impedances. Its application to invert a passive differentiator resulted in a versatile low-frequency differential integrator. Its application to invert a passive RC integrator yielded a versatile low-frequency differential differentiator. Each employs a single time constant, has a resistive input, and a reasonably high $Q$ value. Simulation and experimental results verify the theoretical expectations. The active-network synthesis can be applied to obtain other varied realizations. The differential integrators and differentiators could easily be modified to obtain inverting and non-inverting integrators and differentiators by simply grounding one of the two inputs in each of the differential configurations. Additionally, the limited bandwidths of the circuits mitigate the contribution of the noise and yield output waveforms with large signal to noise ratios.

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References