The Finite Volume Method

Chapter 4
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MECH 663
The General Scalar Equation

\[
\frac{\partial (\rho \phi)}{\partial t} + \nabla \cdot (\rho u \phi) = \nabla \cdot (\Gamma \nabla \phi) + Q_\phi(\phi)
\]

The scalar equation is a balance equation written in differential form.
Balance Form

To recover the balance form we integrate over some CV

\[
\int \frac{\partial (\rho \phi)}{\partial t} dV + \oint (\rho \mathbf{u} \phi) \cdot d\mathbf{S} = \oint (\Gamma \nabla \phi) \cdot d\mathbf{S} + \int Q(\phi) dV
\]

\[
\int \frac{\partial (\rho \phi)}{\partial t} dV + \sum_{f=\text{nb}(V)} \left( \int (\rho \mathbf{u} \phi) \cdot dS \right) - \sum_{f=\text{nb}(V)} \left( \int (\Gamma \nabla \phi) \cdot dS \right) = \int Q(\phi) dV
\]

\[
\int \frac{\partial (\rho \phi)}{\partial t} dV_p + \sum_{f=\text{nb}(V_p)} \left( \int J \cdot dS \right) = \int Q(\phi) dV_p
\]

\[
J = J^C + J^D
\]

\[
= (\rho \mathbf{u} \phi) - (\Gamma \nabla \phi)
\]
Flux Integration

\[ \sum_{f=nb(V_P)} \left( \int (\rho u \phi) \cdot dS \right) - \sum_{f=nb(V_P)} \left( \int (\Gamma \nabla \phi) \cdot dS \right) \]

\[ \int J \cdot dS = \sum_{ip(f)} (J \cdot S)_{ip} = \sum_{ip(f)} J_{ip} \cdot w_{ip} S_f \]

\[ (\rho u \phi)_f \cdot S_f = (\rho u \phi)_{1(f)} \cdot w_{1(f)} S_f + (\rho u \phi)_{2(f)} \cdot w_{2(f)} S_f \]
**Volume Integration**

\[
\int_{V_P} \frac{\partial (\rho \phi)}{\partial t} \, dV = \int_{V_P} Q(\phi) \, dV_P = \sum_{iP(P)} (Q_{iP} V_{iP})
\]

\[
Q_P V_P
\]

1 point gauss integration

4 point gauss integration

9 point gauss integration

\[
Q_{1(P)} V_{1(P)} + Q_{2(P)} V_{2(P)} + Q_{3(P)} V_{3(P)} + Q_{4(P)} V_{4(P)}
\]
Semi-Discretized Equation

\[
\sum_{f = \text{nb}(V_p)} \sum_{ip \sim ip(f)} \left( w_{ip} (\rho u \phi)_{ip} \cdot S_f \right) - \sum_{f \sim \text{faces}(V_p)} \sum_{ip \sim ip(f)} \left( w_{ip} (\nabla \phi)_{ip} \cdot S_f \right) = \sum_{ip \sim ip(V_p)} w_{ip} Q_{ip} \Omega_p
\]
BC: Flux Specified

\[ (\rho u \phi - \Gamma \nabla \phi)_b \cdot \mathbf{S}_b = (-\Gamma \nabla \phi)_b \cdot \mathbf{n}_b S_b = q_{b,\text{specified}} S_b \]

\[ \mathbf{u}_b = 0 \]
BC: Value Specified

\[ \phi_b = \phi_{b,\text{specified}} \]

No Diffusion

\[ J_b \cdot S_b = (\rho u \phi)_b \cdot S_b = (\rho_b u_b \cdot S_b) \phi_{b,\text{specified}} \]

\[ = F_b \phi_{b,\text{specified}} \]
Spatial Variation

\[
\phi(x) = \phi_p + (x - x_p) \cdot (\nabla \phi)_p + \frac{1}{2} (x - x_p)^2 : (\nabla \nabla \phi)_p \\
+ \frac{1}{3!} (x - x_p)^3 : : (\nabla \nabla \nabla \phi)_p + \ldots \\
+ \frac{1}{n!} (x - x_p)^n : : : (\nabla \nabla \nabla \nabla \phi)_p + \ldots
\]

\[
\phi(x) = \phi_p + (x - x_p) \cdot (\nabla \phi)_p + O(\Delta x^2)
\]
Mean Value Theorem

\[ \Phi_p = \frac{1}{\Omega_p} \int (\phi) d\Omega \]

\[ = \frac{1}{\Omega_p} \int \left[ \phi_p + (\mathbf{x} - \mathbf{x}_p) \cdot (\nabla \phi)_p + O(\Delta x^2) \right] d\Omega \]

\[ = \phi_p \int d\Omega + \frac{1}{\Omega_p} \int (\mathbf{x} - \mathbf{x}_p) d\Omega \cdot (\nabla \phi)_p + \frac{1}{\Omega_p} \int O(\Delta x^2) d\Omega \]

\[ = \phi_p + O(\Delta x^2) \]
Problem 3.4

Consider one-dimensional diffusion in the calculation domain shown in the figure below with $\Delta x = 1$ for all control volumes. Assume that the source term $Q=50x$, and that $\Gamma$ is constant with a value $\Gamma = 1$. Let the boundary values of $f$ be $\Phi_0 = 100$ and $\Phi_4 = 500$.

a) Write the discrete equations for each of the cells 1, 2 and 3.

Solve the discrete equation set using Gauss-Seidel iteration and report the resulting cell-centroid values.

Compute the diffusion flux at each of the cell faces $f_0$, $f_{12}$, $f_{23}$ and $f_4$ using the same discretization approximations made in obtaining the discrete equations.

Show that the conservation principle is satisfied on each discrete cell and on the whole domain.

Find the exact solution to this problem. Find the percentage error in the computed solution at each cell centroid.
Problem 3.5

Consider the computational domain in the figure below. Using the conservation of mass principle compute the mass flux at faces $f_{12}$, $f_{23}$ and $f_4$.

Write the steady state transport equation for a scalar, $\Phi$, being advected through the domain with no diffusion and no source term

a. Write the discrete equations for cells 1, 2 and 3 (Use the upwind profile),

b. If we assume that $(d\Phi/dx)_4 = 0$ compute the value of $\Phi_4$.

c. If $(\Delta x=1)$ and $(\Delta y=0.5)$ compute the values of $\Phi_1$, $\Phi_2$, $\Phi_3$ and $\Phi_4$. 
Problem 3.6

Consider the computational domain shown below. Write the diffusion equation for this domain for $\Gamma=1$ and $Q=0$
Write discrete equations for cells 1, 2 and 3
Can you find a unique solution for $\Phi_1$, $\Phi_2$ and $\Phi_3$. Explain?